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EQUATIONS.*

By James M. Taylor.

The Axioms of Equals not Applicable to Equations.

The statement that two expression denote the same number is called an equality. This name is precise and descriptive; equally so are identity and equation as names respectively of an unconditional equality and a conditional equality. The beginner in algebra should clearly distinguish between these two kinds of equalities.

Some of the characteristic differences between equations and identities are the following:

The *numbers* in an *equation* are classified as *knows* and *unknows*.

An equation is a conditional equality, *i. e.,* it holds true only for a limited number of values of its unknown letters.

An equation is to be solved, *i. e.,* we are to find the values of its unknown or unknowns.

In solving equations we use

The numbers in an identity are not classified as knows and unknowns.

An identity is an unconditional equality, *i. e.,* it holds true for all values of its letters, or all values between certain limits.

An identity is to be proved, *i. e.,* we are to prove that its members are identical expressions.

In proving identities we use

the principles of equivalency.	the axioms of equals, or the principles of identical expressions.
Equations are chiefly useful in stating and solving problems.	Identities are chiefly useful in transforming and solving equations.

The pupil has proved identities and used them in his first lessons in arithmetic. Counting two implies the identity $1 + 1 = 2$. In proving identities we use the axioms of equals, or preferably the axiomatic principles of identical expressions. The latter emphasize what must be maintained, identical expressions in the two members.

In solving an equation we seek the values, or sets of values, of its unknowns which will render the equation an identity. Some equations are so simple that we discover their solutions by inspection. But, in general, to solve an equation we must derive from it simpler equations. That any derived equation may be of any use, we must know that it is equivalent to the given equation, or at least that its solutions include all those of the given equation. By the axioms of equals we can not determine whether any derived equation fulfills this condition or not. To illustrate this fact, let us consider some equations derived by applying these axioms to equations.

Suppose we have given the equation

$$1 - \frac{x^2}{x-1} = \frac{1}{1-x} - 6. \qquad (1)$$

Multiplying the conditionally equal members of (1) by $x - 1$, and transposing we obtain

$$x^2 - 7x + 6 = 0. \qquad (2)$$
$$\therefore x = 1 \text{ or } 6.$$

If the axioms proved equivalency, we would conclude that equations (1) and (2) were equivalent. But the derived equation (2) is not equivalent to (1); for $x = 1$ does not satisfy (1). But $x = 6$ does satisfy (1); hence the solutions of the derived equation (2) include one solution of (1). But as one solution was introduced, the question arises whether any solution was *lost* in the operation. To answer this question and to

explain how the new solution was introduced, we need the doctrine of equivalency.

As another illustration take the two equations,

$$x - 1 = 2, \tag{3}$$

and

$$x + 1 = 5. \tag{4}$$

Multiplying the members of (3) by the corresponding members of (4) we obtain

$$x^2 - 1 = 10, \quad \text{or} \quad x = \pm\sqrt{11} \tag{5}$$

The solution of equation (3) is 3 and that of (4) is 4, while the solutions of the derived equation (5) are $\pm\sqrt{11}$. Hence by multiplying the conditionally equal members of (3) by those of (4) we lost both the solution of (3) and that of (4) and introduced the two solutions of (5).

Dividing the members of (3) by those of (4), we obtain

$$\frac{x-1}{x+1} = \frac{2}{5}, \quad \text{or} \quad x = \frac{7}{3}. \tag{6}$$

Here the solutions of (3) and (4) were lost and the solution of (6) was introduced.

Adding the members of (3) to those of (4) we obtain

$$2x = 7, \quad \text{or} \quad x = 7/2. \tag{7}$$

Here the solutions of (3) and (4) were lost, and the solution of (7) was introduced.

Subtracting the members of (3) from those of (4) we obtain

$$0x + 2 = 3, \quad \text{or} \quad 0x = 1, \tag{8}$$

which has no solution, or is impossible.

In this case the solutions of (3) and (4) were lost and no solution was gained.

Again from (3) and (4) we obtain,

$$x + 1 = 4, \tag{9}$$

and

$$x - 1 = 3. \tag{10}$$

Multiplying the members of (9) by those of (10) we obtain

$$x^2 - 1 = 12, \quad \text{or} \quad x = \pm\sqrt{13}. \tag{11}$$

Here the two solutions of (9) and (10) were lost and the two solutions of (11) were introduced.

Dividing the members of (9) by those of (10) we obtain

$$\frac{x+1}{x-1}=\frac{4}{3}, \quad \text{or} \quad x=7. \tag{12}$$

In this case the two solutions of (9) and (10) were lost, and the solution of (12) was introduced.

Adding the members of (9) to those of (10) we obtain

$$2x=7, \quad \text{or} \quad x=7/2. \tag{13}$$

Here the two solutions of (9) and (10) were lost, and the solution 7/2 was introduced.

By subtracting the members of (9) from those of (10) we obtain the impossible equality, $0x-2=-1$, or $0x=1$. Writing the equations in the form

$$x-3=0, \tag{14}$$

and

$$x-4=0, \tag{15}$$

and then multiplying as above we obtain

$$(x-3)(x-4)=0. \tag{16}$$

The derived equation (16) is equivalent to the two equations (14) and (15) jointly; hence no solution was lost or gained. But if we change the order of the members of (14) and then multiply we obtain the identity $0=0$.

Adding the members of (14) to those of (15) we obtain

$$x=7/2.$$

Subtracting the members of (14) from those of (15) we obtain the impossible equation $0x=1$.

Of the operations above, the doctrine of equivalency of equations would allow none except the first multiplication with equations (14) and (15), by which no solution was lost or gained.

It might be objected that the x's in the equations above denoted different numbers. But let us take two equations which are equivalent; *e. g.*,

$$x - 1 = 2, \tag{17}$$

and

$$x - 1 = 2. \tag{18}$$

Multiplying the members of (17) by those of (18) we obtain

$$x^2 - 2x + 1 = 4, \quad \therefore x = 3 \text{ or } -1. \tag{19}$$

Here one solution was lost and one introduced.

Dividing the members of (17) by those of (18) we obtain the identity $1 = 1$.

Subtracting the members of (17) from those of (18) we obtain the identity $0 = 0$.

Changing the order of the members of (18) and adding we obtain the identity $x + 1 = x + 1$.

Again squaring the members of the equation

$$-2 + \sqrt{2x + 8} = 2\sqrt{x + 5}, \tag{20}$$

and then squaring again we obtain

$$x^2 - 16 = 0. \quad \text{or} \quad x = \pm 4. \tag{21}$$

But neither $+4$ nor -4 will render (1) an identity. Hence both solutions of (21) were introduced by squaring. The axioms give no hint of this fact. In fact until very recently, both of the solutions of (21) would have been given by text-book writers as solutions of equation (20).

The doctrine of equivalency proves that if equation (20) has any solution, it is included among the solutions of (21). Hence (20) is to be tested for each solution of (21). These cases will suffice to illustrate that by the axioms of equals, we can not prove what we want to know concerning any derived equation. The question arises, "Are the axioms of equals ever applicable to equations? For an answer we have not far to look; the answer is in the meaning of the word *equals*. In the axioms, equals, or equal numbers, mean unconditionally equal numbers. The members of an identity denote such numbers. Hence these axioms are applicable to identities. But the members of an equation do not denote unconditionally equal numbers. Hence these axioms are not applicable to equations.

To be applicable to equations the axioms must still be true when in them the phrase "conditionally equal" is substituted for the word "equal." Making this substitution in the axiom, "If each of two numbers is equal to the same number, they are equal to each other," we have.

"If each of two numbers is conditionally equal to the same number they are conditionally equal to each other."

This does not sound like an axiom, nor does it act like one. In fact it is not true.

E. g., if we take the two equations,

$$x - 1 = 3 \quad \text{and} \quad x - 5 = 3,$$

and apply this proposition as an axiom we obtain

$$x - 1 = x - 5, \quad \text{or} \quad 0x = 4,$$

which is impossible for any value of x.

Thus we see that the futile results in equalities (5), (6), (7), (8) are obtained from (3) and (4) not by what the axioms of equals really say, but by what they are slanderously charged with saying.

If in the equations (3), (4), (9), (10), (14), (15), (17), (18), the conditional element were removed by substituting for each unknown its value, then the results obtained by each operation in accord with an axiom of equals would be an identity.

Thus if any number of operations are performed on the members of an identity or of identities and each is in accord with some axiom of equals, the derived equality will be an identity.

But if any number of operations are performed on the members of an equation or of equations, each of which is admissable on identities the derived equality may or may not be an equation equivalent to the given equation or equations.

Hence from the fact that all the operations performed on an equation or equations are admissable on identities, our futile conclusion is that the derived equality *may* or *may not* be an equation equivalent to the given equation or equations.

Thus when a pupil gives an axiom of equals as a proof of equivalency he gives a false reason, which is a thousand times worse than no reason at all.

If any reason is given for the equivalency of two equations, let it be a principle of equivalency. The objection that the beginner can not understand the principles of equivalency and prove them is not well founded. The first principles of equivalency are very simple in thought, and just as easy of application as the axioms. Their general proof should not be required of the beginner, but he should often verify them in particular examples, and thus make their thought familiar and convince himself of their truth. The word equivalent or equivalency should not be used at first. Each principle should be illustrated, and then stated, for equations in one unknown, somewhat as follows:

I. If two expressions are identical and either is substituted for the other in an equation, the unknown has the same values in the derived equation as in the given one.

II. If identical expressions are added to or subtracted from both members of an equation, the unknown has the same values in the derived equation as in the given one.

III. If both members of an equation are multiplied or divided by the same known expression, not denoting zero, the unknown has the same values in the derived equation as in the given one.

This form of statement has the merit of emphasizing what the pupil's attention should be fixed upon, viz., the values of the unknown. These values are the game he is seeking and his eye should be kept upon them. Only the doctrine of equivalency can keep this game in sight.

The Direct Problem of Making Equations.

The natural, common-sense and pedagogical order is to solve and to study first the direct problem and afterwards the inverse problem. Addition comes before subtraction, multiplication before division or factoring, and addition of fractions before decomposition of fractions. We learn how to make differential equations before we try to solve them. According to this fundamental principle, the direct problem of making quadratic and higher equations should precede the inverse problem of solving them. But in equations the common practice is to ignore the direct problem and to take up first the difficult inverse problem. This neglect of the direct problem is the more to be regretted, because the simple problem of making

equations which have given roots brings clearly into view the fundamental principles and the fundamental method of solving equations and also the nature of this inverse problem. Moreover the study of this simple direct problem makes clear some of the most important properties of equations, such as, the number of its roots, the relation of its roots to its coefficients, the appearance of surd, imaginary and complex roots in conjugate pairs, etc.

Having given pairs of particular numbers the pupil should make quadratic equations which have these numbers as roots. At first the given roots should be real and commensurable, then conjugate surds, then conjugate imaginaries and then conjugate complex numbers. The pupil should carefully note that, in each of the above cases, the coefficients of each equation are real and commensurable. Then let equations be made which have such pairs of roots as are not included in the cases above, and have the pupil note the fact that in each equation the coefficients are not all real and commensurable. The pupil is thus prepared for solving and studying the general problem below:

To find the equation whose solutions are a_1 and a_2 where a_1 and a_2 are any numbers whatever.

The equation whose solution is a_1 is

$$x - a_1 = 0. \tag{22}$$

The equation whose solution is a_2 is

$$x - a_2 = 0. \tag{23}$$

By multiplying we obtain

$$(x - a_1)(x - a_2) = 0. \tag{24}$$

But we have the identity

$$(x - a_1)(x - a_2) \equiv x^2 - (a_1 + a_2)\,x + a_1a_2. \tag{25}$$

From equation (24) by identity (25) we obtain

$$x^2 - (a_1 + a_2)x + a_1a_2 = 0. \tag{26}$$

Equation (26) is equivalent to (24) and therefore to (22) and (23) jointly.

Some of the obvious conclusions from this problem are the following corollaries:

COR. 1. From equation (26) it follows that every equation that has two given roots is a quadratic equation.

Again from the identity (27)

$$\begin{aligned} x^2 + 2px + q &\equiv (x+p)^2 - (p^2 - q) \\ &\equiv (x + p + \sqrt{p^2 - q})(x + p - \sqrt{p^2 - q}), \quad (27) \end{aligned}$$

it follows that every quadratic equation can be put in the form of (24).

Hence conversely every quadratic equation has two roots and only two roots.

A glance at the solution above would lead the pupil to infer that the degree of any equation and the number of its roots are equal.

COR. 2. From equation (26) we see that when in a quadratic equation the coefficient of x^2 is $+1$, the coefficient of x is minus the sum of the roots, and the known term is the product of the roots.

COR. 3. In equation (26) the coefficients $-(a_1 + a_2)$ and a_1a_2 are real and commensurable when the roots a_1 and a_2 are both real and commensurable, or when they are conjugate surds, conjugate imaginaries, or conjugate complex numbers, and in no other case.

Hence conversely when the coefficients of a quadratic equation are real and commensurable, its two roots are real and commensurable, or they are conjugate surds, conjugate imaginaries, or conjugate complex numbers.

From Cor. 2 or equation (26), the equation which has any two given roots can be written out, and thus the relation between the roots and the coefficients of an equation made familiar.

To solve the general quadratic equation (26) we retrace our steps to the linear equations (22) and (23). These equations, or the values of x given in them, are the solutions of (26). We thus see that the solving of an equation involves the same principles of equivalency as its making and that where the making involves finding the product of given linear factors the solving involves finding the linear factors of a given product.

Thus the pupil sees that factoring is not one of several ways

of solving a quadratic or higher equation, but that fundamentally factoring is the *only* way of solving such equations. He can not be too strongly impressed with this fact, so that whatever devices he may later employ in solving such equations he will recognize them as devices for factoring.

The great gain in first studying the direct problem in equations is not that some properties of equations are thus more easily proved, but that these properties are thus made more evident, tangible, real and familiar to the pupil at the very beginning of his study of quadratics and that from the first he has a clear idea of his problem and the fundamental method of solving it.

Substitution and Factoring in Solving Systems.

The simplest system of n equations is one of the form of system (A),

$$\left.\begin{aligned} x_1 &= a_1 \ (1) \\ x_2 &= a_2 \ (2) \\ &\cdots\cdots\cdots\cdots \\ x_n &= a_n \ (n) \end{aligned}\right\} \qquad (A)$$

in which each of its n equations contains but one unknown and is solved for that unknown. To indicate that two or more equations are simultaneous and form a system they should be joined by a brace as in (A).

To solve a system of n linear equations in n unknowns is to find a system of form (A) which is equivalent to the given system. To find such a system we may proceed as follows: Solve for x any equation of the system, say the first, and substitute this value of x in each of the other equations, then in the second system x will not appear in any equation after the first. In this second system solve any equation after the first, say the second, for y, and substitute this value for y in each of the other equations, then in the third system y will not appear in the first equation, and neither x nor y will appear in any equation after the second. Continuing this operation there will be obtained finally a system of the form of (A). All the systems obtained as above will be equivalent. Hence the last system will be the solution of the first.

The operation outlined above is called elimination by substitution. The principle of substitution upon which it is based

should be stated at the outset, but at first no proof need or should be given. All the other so-called methods of elimination are but modified forms of this fundamental method by substitution. *E. g.*, elimination by comparison is clearly a case of substitution. Elimination by addition or subtraction is simply a way of substituting for certain multiples of the unknown in one equation, the values of these multiples as obtained from the other equations of the system. Elimination by indeterminate coefficients is a modified form of elimination by addition. Elimination by division is simply a way of substituting in one equation of the system, the value of a combination of some of its unknowns as obtained from another equation of the system.

Since the principle of substitution underlies all the so-called different methods of elimination, the method of substitution should be taught first; for only thus will the pupil gain at the outset, a clear idea of the fundamental principle underlying all the methods, and see clearly the relation of the other methods to this fundamental one.

But the common practice in our teaching is to present first of all the so-called method of elimination by addition. This blinds the student to the one principle underlying all his work in elimination and starts him on his career of working without thinking in the solution of systems.

If a system contains one or more equations above the first degree, and each of them can be written in the type-form, "The product of two or more linear factors equal to zero," it is easy to write the linear systems which jointly are equivalent to the given system. If a system contains one or more equations which are above the first degree, and these equations can not be factored, then the first problem is to find by sustitution a new system equivalent to the given system, whose equations above the first degree can be factored. This being done we can write and solve the linear systems which jointly are equivalent to the given system. Thus we see that factoring is not only the fundamental method of solving quadratic and higher equations in one unknown, but it is the fundamental method of solving systems which involve equations above the first degree.

In the study of equations and systems therefore we would emphasize the following points:

1. Do not misuse the axioms of equals by applying them to equations.

2. Make clear and familiar the fundamental principles of equivalency of equations and systems.

3. Discover the fundamental properties of quadratic and higher equations by solving and studying the direct problem of making such equations.

4. Emphasize that to solve an equation or a system above the first degree we must first find the linear equations or linear systems which jointly are equivalent to the given equation or system, and that fundamentally the only way of deriving these linear equations or systems is by factoring.

5. In solving systems use first the method of elimination by substitution and as each of the other methods is employed show that it is but a convenient mode of effecting substitutions.

Thus we would give our pupils a better chance of becoming thinking workers in mathematics. Our problem in teaching mathematics to-day is not so much to make the courses easy for the unthinking, as it is to make clear the fundamental principles of the science, and to stimulate our pupils to grasp, to prove and to use these principles.

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SHORTENING THE COURSE IN ARITHMETIC.*

By Sherman Williams.

The great amount of time given to the study of arithmetic, and the very meager results secured, is not after all due so much to the extent and variety of the work attempted as to inefficient teaching, though much of the work might be omitted without serious loss. All the work in arithmetic might be included under three heads.

1. *Memorizing.*—Under this head would come the learning of the tables of addition, multiplication, tables of denominate numbers, the aliquot parts of a dollar and the like. All this work, and a considerable other work in arithmetic could be done in the first four years of school without taking, on the average, more than ten or fifteen minutes a day.

2. *Acquiring Processes.*—This work should be completed by the end of the sixth year, and not more than twenty minutes a day need be given to the work during the fifth and sixth years of school life. The acquisition of processes should be chiefly memory work, reasons should not be required or given.

3. *A Study of the Principles Underlying the Processes.*—This is really high school work though something can be done in the seventh and eighth years. At the close of the sixth year the child should have a good working knowledge of arithmetic, sufficient to serve him in the common affairs of life. The later work, if undertaken, is disciplinary and does not help greatly in the application of arithmetic to the affairs of everyday life.

Suppose we consider separately the three phases of work that have been indicated and see if we can discover where mistakes are commonly made and where time can be saved. There are a few fundamental principles that must constantly be kept in mind.

1. Children like to do the things that they think they can do well.

* Read at the meeting of the Syracuse Section.

2. Progress is easier and more rapid when the separate steps are short.

3. Only one step should be taken at a time.

4. Things that need to be permanently known must be fixed through a great deal of drill that extends over a considerable period of time.

5. Confidence is a great source of inspiration and power and should be judiciously cultivated.

6. Nothing that one thoroughly understands is difficult.

7. Children will accomplish little or nothing unless they are interested.

8. There can be no interest without attention.

9. No child can give attention for a long period of time.

For the first four years of school life no period of recitation should exceed fifteen minutes—ten minutes would be far better. Nowhere below the high school should the recitation period exceed twenty-five minutes, and the subject must be pretty interesting, and the teacher pretty skilful to hold the attention of any class for twenty-five minutes.

There is comparatively little difficulty in teaching children to write and read numbers. The first trouble is in teaching children to add, yet it is one of the easiest things to teach children to add so that they will never make a mistake through ignorance—they may through carelessness. No one ever makes a mistake in adding two and two. This is one combination. There are only forty-five, and one is no more difficult than another, and a child who learns one combination perfectly can learn them all perfectly, and yet mistakes in adding are very common. No one ever really adds. One sees or hears 3 and 5 and recalls 8. It is a case of purely arbitrary memory. Any attempt to develop the addition table through the use of objects is foredoomed to failure. It is not learned that way and cannot be no matter how skilful the teaching, or how earnest the efforts of the pupils. The use of objects in teaching addition is rarely if ever helpful, and very commonly harmful. It very often leads to the pernicious finger-counting habit, which is so difficult to overcome.

Children make mistakes in adding simply because of inefficient teaching. They are drilled upon the combinations which

they already know perfectly just as much as upon those which they do not know. They are not trained to have confidence in themselves. To some extent they are trained to not have confidence. Often they are told to add up a column of figures, and then to be sure that they are right add the column downwards. Often the two additions will not agree and confidence is gone. Children should not be told when they have made a mistake in adding. The teacher should observe what combination it is that the child does not know and then give much drill upon it.

There should be some system in the work in addition. It is not a bad plan to take first those combinations the sum of which does not exceed nine. These constitute twenty of the forty-five combinations. It will be found that there will be most difficulty with the 6s, 7s, 8s and 9s. There should be most drill upon these.

There are many devices for drill in addition. Counting to 100 by 1s, 2s, 3s, 4s, 5s, 6s 7s, 8s and 9s. Counting both forward and backward and beginning with 1, 2, 3, 4, 5, 6, 7, 8 or 9, and in counting backward beginning with 100, 99, 98, 97, 96, 95, 94, 93, 92 or 91. Counting by series, both direct and broken. Sometimes a serious mistake is made by asking the child when adding a column of figures to notice that 4 and 6 are ten, 7 and 3 are ten, etc., and to take the tens separately. This is giving drill upon the combinations well known, and therefore easy, rather than upon the combinations less well known—that is putting the drill where it is least needed.

Never allow a child to guess at an answer. He either knows it or he does not. It is a case of arbitrary memory. It is not a matter that can be thought out. He is as well off if he guesses wrong as though he guessed right, possibly better off.

A very effective method of drill is to have the whole class add together silently, first giving the numbers so slowly that the dullest child in the class can do the work, then gradually increasing the speed till the brightest and quickest child is taxed to his utmost. This is work that children like. Every child can do a part of it. Each one tries to keep up with the class a little longer each day.

It is a very easy matter to drill the children so that they all

will have learned all the combinations in addition by the end of the third year. They will have learned much else beside.

It is not well for a child to be constantly meeting things wholly new. This should be carefully guarded against. The subtraction table should be made more or less familiar through the study of the addition table, and in like manner much of division should be learned while studying multiplication, and the study of division should lead to an understanding of fractions. When a child learns that 5 and 4 make 9 he should be required to present that fact in a different way as 5 and what make 9. 4 and what make 9. There should be much of this kind of work in connection with addition, and similar work leading to division in connection with multiplication. The multplication table will be learned in much the same way as the addition table and quite as easily.

Before any written work is attempted there should be much mental work involving addition, subtraction, multiplication, and division. Most processes are most easily learned through mental work. Of course in a strict sense all arithmetical work is mental.

The learning of processes should be mainly a matter of memory. No explanations should be given at the time the processes are learned. That is a work that demands a greater maturity of mind than most children possess at this time. In all the early work in arithmetic a good model should be given and it should be followed perfectly, no variation being allowed. Processes should be fixed in mind by much repetition.

The first written work in addition should consist of examples in which the sum of no column should exceed nine. Keep up practice with such examples till the pupils are both accurate and rapid in their work, then introduce the matter of carrying but do not attempt to explain it. It is quite enough to say that people add that way. That is all the explanation that there is for a great deal of the work that comes up in school. People do so and so. They might do differently but they do not.

If the oral work in addition is well done there will be little trouble with the written work. The same is true of all the fundamental operations.

I see no choice in the two methods in subtraction in the case

in which the number of the subtrahend is larger than the corresponding number of the minuend, but whichever method is chosen should be adhered to up to the seventh grade.

Only one step should be taken at a time, and that the shortest possible one. This principle is violated oftener than almost any other. Take multiplication for example. It is no uncommon thing for a teacher to begin the written work by giving an example in which the multiplier has two figures and the multiplicand three or four. Here are three steps at once instead of one, multiplying a single number, carrying and dealing with partial products. Is it to be wondered at that with work done in this manner many of the pupils become hopelessly confused?

The same kind of a mistake is made in teaching division, and with even more disastrous results.

When the four fundamental rules have been mastered there is little else to be learned in arithmetic. There is merely a difference in form and terminology, or nomenclature. There are no new principles.

After the completion of the fundamental rules and before beginning the work in fractions the pupils should be led to see clearly that multiplying or dividing both divisor and dividend by the same number does not change the value of the quotient. It is not enough that they be shown this and that they are able to state the fact glibly, but they must actually see it as the result of many problems worked by them, the work being carried over a considerable interval of time. The clear understanding of the work in fractions depends upon a perfect understanding of this principle. Two great difficulties in the teaching of arithmetic are, first, that the pupils pass on to more advanced work before they really fully understand the fundamental operations. Second, that teachers make difficulties for them by telling them they are to take up something new when in point of fact no new principles are involved, and in addition to this they make several cases as in fractions, percentage, etc., when the whole work should be regarded as a unit with pupils below the seventh grade.

A fraction is simply an indicated division expressed in a way new to the children, and the children should be told this at the

outset. So decimal fractions are merely another form of expressing indicated division. It is a pretty good plan to make the children quite familiar with decimals while teaching the fundamental rules, without saying anything about their being decimals, or differing in any way from the numbers they have been dealing with save in the matter of value. There really is no other difference.

Children are often taught that they are entering upon a new and different field when they begin the study of fractions, but of course this is not true, and they ought not be allowed to think there is anything new save form and nomenclature. They have been accustomed to add concrete things of the same kind and know that they cannot add unlike things. The same should be seen to be true in fractions. They have added apples to apples, and pears to pears, but not apples to pears. So they may add fourths to fourths and fifths to fifths, but not fourths to fifths.

What is to be done with such numbers. A fraction is always an indicated division. What they called the dividend in whole numbers they call the numerator in fractions, and what was known as the divisor in whole numbers is called the denominator in fractions. The children know that in any individual case they may multiply both divisor and dividend by the same number without changing the value of the expression. So with numerator and denominator. The fifths may be made twentieths by multiplying both numerator and denominator by the number four. The fourths may be made twentieths by multiplying both numerator and denominator by five, and twentieths can be added. With this matter clear there is no further trouble with fractions, or at any rate any fractions that may properly be considered before the seventh year, if you proceed slowly, taking only one step at a time. There is no need of any plan of finding the least common multiple. That can be seen by inspection. It is always the largest denominator or some number of times that number. There should be a great amount of mental work in fractions. All kinds of work in fractions should be made clear through simple mental work.

During the fifth and sixth years there should be a great number of practical examples dealing with real conditions, and covering a wide range of work. There should be many simple

examples in mensuration, and each example may be made to cover more than one point after the single steps are clear, for example a bin is so long, wide, and deep. How many cubic feet does it contain? How many bushels will it hold, how many barrels, how many pounds of wheat? Have pupils measure things about them, piles of wood, floor space, walls, etc. Then have them work with measurements given them, but in all cases have them substitute drawings or diagrams for the real thing, continuing that practice till they have formed the habit of visualizing.

The whole range of work in percentage can be made perfectly clear through oral work. Have much drill in such examples as these:

6 is what per cent. of 18?

What per cent. of 20 it 5?

1/3 of 18 is what per cent of it?

5 per cent. of 20 is what per cent of 10?

12 per cent. of 200 is 10 per cent. of what number?

In teaching percentage, as in teaching fractions, difficulties are created by making many different cases where there is really only one. Of course there are minor differences that you may recognize in dealing with older children, but with the younger ones it merely serves to confuse.

I have outlined in a general way the work that can easily be completed by the end of the sixth year, and done well and thoroughly without the expenditure of any very great amount of time. It will give the pupils all the knowledge of arithmetic that ninety-nine out of a hundred will ever have any occasion to use. In my judgment the study of arithmetic should not be carried beyond what I have indicated until the latter part of the high school course. If one wishes something may be done in the seventh and eighth grades, but it seems to me that the time can be used to a much better advantage with other subjects.

So far I have said nothing about omissions save by implication. I am loath to attempt it. There is sure to be no general agreement upon this point. An argument can be made for the retention of any given topic, or for the introduction of almost any. They can be shown to be of some value. But the real question is not, is this or that of value, but is it at this point in the school course of greater value than some other subjects.

In teaching any subject, in any grade, the question is not is it of value, and can the work be done, but can it be done at a greater profit to the pupils than some other work.

With these thoughts in mind I will venture to express the belief that in the first six years of school life all examples should consist of small numbers and be expressed in the simplest way, and that the greater part of the examples should be those that can be solved mentally. I believe that little or nothing should be taught in regard to the least common multiple or the greatest common divisor. I believe that absolutely no work should be required in proportion, compound partnership, partial payments, true discount, average of accounts, arithmetical progression, geometrical progression, square or cube root.

With the elimination of these subjects and simple treatment of the others there is no reason why wholly satisfactory work may not be done with a small expenditure of time, and the work completed at the end of the sixth year. This can be done in any school where there is a competent teacher, as well in the most remote rural school as in the best village or city school.

GLENS FALLS, N. Y.

"If you and I—just you and I—
Should laugh instead of worry;
If we should grow—just you and I—
Kinder and sweeter hearted,
Perhaps in some near by-and-by
A good time might get started;
Then what a happy world 'twould be
For you and me—for you and me."

MATHEMATICS IN THE ETHICAL CULTURE HIGH SCHOOL.

By C. B. Walsh.

In giving a report of the mathematics in the Ethical Culture High School I do so with the fear that we of the school may seem to profess or at least to consider that we have discovered or attained the ideal. Let me preface the paper by saying that we are keenly conscious that there are defects both in the conception and in the execution of our course. Whatever I may say that sounds otherwise please pardon on the ground of enthusiasm. The purpose I hold in mind in giving this paper is an exchange of testimony—a practice which, to my way of thinking, is very helpful in all professions. Hence, I shall feel gratified if my paper simply provokes further discussion of the nature indicated. This exchange of testimony I consider a very encouraging sign of our generation. For example of what I mean look at the lists of problems found effective by various teachers now being published in *School Science and Mathematics.*

Moreover, in this paper I am going to deal with concrete details. It does not seem to me profitable to dwell on general principles before a group of professional teachers. You have long been familiar with such principles.

Let me group the reports under four main heads:

I. Aims of the Course.
II. Organization and Equipment of the Department.
III. Course of Study.
IV. Pupils' activities.

I. Aims.

The aims of our high school course are set forth in our pamphlet of the "Course of Study in Mathematics" better than I can state them. Hence, I shall merely quote from the same.

1. To furnish a solid foundation on which the more advanced work of college and technical school may be based.

2. To train pupils in habits of attention, accuracy and system, in dealing with mathematical material.

3. To develop an appreciation of the methods and spirit of pure science.

4. To give some conception of the importance of mathematical knowledge for understanding natural laws and for applying them to practical affairs.

Our methods of obtaining these aims, I hope, will be apparent from the details of the paper.

II. The Department—Organization and Equipment.

Although consisting of but few members our department is organized with the principal of the high school, Mr. Stark, at its head. It holds meetings about once a month. Twice a year, spring and fall, meetings of the department of the whole school, including elementary grades, are held. Topics of interest to all members are discussed at all meetings. For instance, suggestions for changes of text, reviews of new books, forms of written work, correlation within the department and with other departments, are among the topics discussed.

In accord with the spirit of the school from kindergarten through normal grades the organization is democratic. Any teacher is at liberty to bring up any topic which he or she believes would be of general interest and general profit.

The school provides a library for the department with an annual fund for additions. This library located in one of the mathematics recitation rooms is of free and easy access to the teachers and to pupils with the sanction of teachers.

The books include general reference and advanced books, histories of the subject, text-books, etc. *School Science and Mathematics* finds its place here, too. It is our hope to accumulate a very complete set of text-books used in foreign schools of secondary grade and a beginning of this collection has been made. To such texts pupils are occasionally referred when they are sufficiently conversant with other languages than English to make this a possible and profitable correlation of language work and mathematics.

Teachers find in this library books useful for the advanced study as well as material for daily service. A classified collection of problems, and sample test papers are also kept here.

Additions to this library are made at the requests of teachers.

The department has also started a mathematical museum.

This is still little more than embryonic, but is promising. It includes an old sextant picked up in a pawnshop on the Bowery and many crude instruments fashioned after the devices of our ancestors. These instruments the boys have made in the school shop. Among them may be found a baculus mensorius, pantagraph, astrolade, quadratus, and sector compasses.

It is the custom of our school to give an annual exhibit of its work, and recently at such exhibitions, while the entire work of the institution has been represented, each year one subject has been emphasized. Year before last the work of mathematics was especially exhibited. At that time, by means of exercises of pupils, printed reports, outlines, and summaries the work was displayed so as to show the continuous development from the kindergarten through the high school. The work was arranged both by grades and by topics. That is, besides showing the sequence by classes, groups of papers on graphic work, old instruments, real problems, etc., were shown. This material has been preserved under similar groupings.

A so-called current exhibit is always in place in the building by means of which specimen papers from each course are kept posted on large swinging leaves, and, as renewed, the older papers are put in portfolios so that the work of a course may be shown in continuous form to date at all times.

III. Course of Study.

The course of study covers four years and includes the usual algebra, plane and solid geometry, logarithms, plane and spherical trigonometry.

The first two years' work is required of all pupils graduating from the school. During these years substantial beginnings are made in algebra and geometry. Usually work in algebra to and including quadratics and in geometry to Book IV is accomplished.

The rest of the course is elective. Time is taken at all stages of the work and particularly during the first two years, *i. e.*, the prescribed work, to emphasize the history, general concepts and applications of the subject that those who end their mathematical training before the conclusion of the four years' course may carry away fundamental principles and facts that make for general culture. The elective work is divided when possible

into college and non-college classes. In the former stress is laid on theoretical or, if you will, pure mathematics—the regular college preparatory work. In the non-college classes more attention is given to applications. Constructive work and numerical problems receive special emphasis.

That students who conclude their course before its completion (*i. e.*, at the end of the second or third year) may have as broad an outlook on the subject as possible, mathematics is taught as such—not as arithmetic, algebra, geometry, etc. The so-called parallel method is pursued in teaching the branches of the great science. Emphasis at different periods is, of course, shifted. For example, the first year of high school, algebra receives most attention and during the second year geometry is in the foreground.

We place text-books in the hands of our pupils for two reasons: (1) as reference books and (2) as collections of problems. The method of class work is practically that used under the syllabus method. Pupils keep note-books for addenda. All written work passed in by the pupils is corrected, returned and preserved by the pupil, who is thus furnished at the end of his course with a sort of loose-leaf ledger note-book of the course.

A topic which seems to have received little general attention we include, *i. e.*, the history of mathematics. We do not give a special course in the history of the subject, but in a seemingly desultory way we develop the history along with the subject, often more as a point of view than as a topic. If, for example, a theorem in geometry is developed which has a significant history, that matter is called up.

We introduce the systematic study of demonstrational geometry by a rapid survey of the subject as an art, *i. e.*, tracing the empirical geometry of the Babylonians and Egyptians in its evolution to the geometry as a science among the Greeks. We believe that by pointing out the crude ideas of our remote ancestors, by following the evolution of new facts, by naming the great masters of our science and their discoveries, we can teach our pupils what general procedure to avoid and what to follow, and emphasize the fact that there is no "royal road" to mathematics; that its development is the result of much

labor; that the science is growing and hence is alive. Such facts tend to enlist the sympathy of the students with the true spirit of science.

As a device for fostering the individuality of pupils, we not infrequently have them develop and present to the class original theorems and proofs. We require pupils at times to look up special topics and report—as, for example, the various historical proof of the Pythagorean proposition. Occasionally for this work it is necessary for the pupil to use a French or a German book. This is a helpful correlation of subjects.

Frequent summaries and outlines are required. The ability to collect and systematize information is a scientific instinct which it is well to foster. Perhaps it may be a question of summarizing a month's or a semester's work. Perhaps it may be important to classify equations and epitomize the method of solution.

A feature of our work which we believe very practical is the field and general work with instruments. After all one of the chief functions of mathematics is measurements—be it of land or of time. In the grade school measurements of distances with the steel tape in and out of the building is begun. In the high school, as time permits, field work in measurements of distances and areas is practiced. The school possesses an excellent transit which is used by the older pupils for such purposes, but before that stage is reached the tape and rod are used. Besides, use is made of the crude instruments, now obsolete or at least obsolescent. I have referred to these instruments earlier in the paper and, hence, shall not dwell on them here—except to say that they prove especially effective for this work for the reason that, devoid of the elaborate mechanism found in modern instruments for accurate adjustments, they illustrate clearly the application of geometrical principles.

Throughout the high school course we aim to make prominent the applications of mathematics to practical affairs. Our chief device in this direction is the real problem. We have a collection similar to that published by Horace Mann School in *Teachers College Record* for March, 1909.

In the early part of the algebra formulas from geometry, physics, engineering, etc., form the basis of this work.

Permit a few illustrations.

Given the fact that the sag and length of trolley-wires are related by the formulas, $S = wl^2/8t$ and $l' = l + 8S^2/3l$ where S = sag in feet, w = weight of wire in lbs. per ft., l = distance in feet between supports, t = tension of wire in lbs., and l' = actual length of wire between supports in feet.

Pupils are required to transform these formulas in various ways, to combine them, and to evaluate them under different sets of data and to tabulate the results of evaluation.

Again, the students are told that a sinking fund is a sum of money set aside annually at compound interest to liquidate a debt. C = number of dollars debt, r = rate of interest, S = sum set aside, n = number of years, we have the relation,

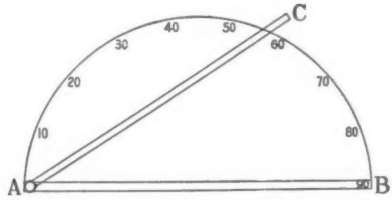
$$C = \frac{S[(1+r)^n - 1]}{r}.$$

From this determine what sum must be set aside annually as a sinking fund by a government owing $500,000 to liquidate its debt at the end of four years, money yielding 5 per cent.

Again—the time of one vibration of a pendulum is given by the formula $t = 16\sqrt{l}$, where t is measured in seconds, and l, the length of the pendulum, is measured in inches. Make a graph showing the relation between the length of a pendulum and its time of vibration. From the graph determine the length of a pendulum which will beat seconds and the time of vibration of a pendulum 3 feet long. Check by calculation.

The work with the instruments described gives rise to various problems, sometimes merely illustrations of geometrical facts, for example:

The following is a diagram of an instrument used for measuring the altitude of the sun or of a star. AB is held hori-



zontally and the pivoted rule AC is aimed at the sun or star. Show that the semicircular scale must be divided into 90 equal parts in order that the edge of the rule may indicate the required number of degrees of altitude.

Again—to measure the height of a church spire, a rod 10 feet long is planted vertically at position *A*, then at position *B*. The observer takes such a position that the top of the spire, the top of the rod, and his eye, are all in line when he stands erect. He measures the distance from this position to the rod, doing this for each of the two positions of the rod, obtaining the measurements 4 feet and 8.3 feet. He also measures the distance between his two positions, finding it to be 138 feet. If the observer is 6 feet tall what is the height of the spire? Explain your solution.

We have groups of problems of this nature for various topics and also lists which are frankly miscellaneous. Again, to be concrete, I quote from a list on similar triangles.

To determine the distance of an island off shore, a stake is set on shore at the nearest point to the island. A line is laid off on the shore at right angles to the line from the stake to the island. At any convenient point *C* in the second line, a third line is laid off at right angles to the line from *C* to the island. Then a stake *D* is set in line with the first and third lines. What distances must be measured? Assume distances and calculate results.

We also have a few problems that lend themselves more readily to graphic solution than to any other means.

Take one illustration. About the time that the Pacific Railroad was opened the newspapers passed around the following question: Suppose that it take a train just one week to run the whole length of the road, and that one train leaves each end of the road each morning. How many trains will a person meet in going the length of the road, not counting the train which arrives as he starts nor the one that starts as he arrives?

Before concluding this part of the paper I want just to touch on two topics which doubtless occur to many here: viz., (1) our method of dealing with "limits" and (2) our use of the graph. On both these points we seek to take a natural, middle course. With regard to limits perhaps a cursory suggestion will be sufficient to indicate our procedure. We deal with the topic on the basis of successive approximations. That is to say, in dealing with the "incommensurable cases" we show that by continuing to change our unit of measurement for one

smaller we can get a negligible remainder and, hence, for all practical purposes, *i. e.*, to any degree of accuracy desired or demanded we can establish the proposition.

The graph we use whenever its use serves to make the problem in question clear or when, as in the illustration already cited, the graphic method of solution is the simplest. We avoid, however, bringing in the topic for its own sake. We use graphs, then, as illustrations, when it is actually desirable in a problem, or when approaching an old problem from a new point of view such as the graph offers, helps to enforce the idea.

Again—before leaving considerations regarding the course of study, I might append a note concerning our grading system. We have half-year classes, so that a pupil failing or entering in the middle of the year need not go back a whole year in the work but has the advantage of a flexible system. Frequently, too, we have two divisions of the same class, into a brighter and a duller section—although, of course, not so labelled.

We have, then considered the aims of our department, its organization and equipment and the course of study. Let us now glance at the students' activities not directly in the course.

IV. Students' Activities.

We have a mathematics club. This is a student organization meeting fortnightly for the consideration of topics which might be classified as "mathematical recreations" and various topics not possible or desirable to introduce into the regular course. The organization is small but interested. Our subject is not one to attract the masses, but is one whose followers have always been devoted. A plan is on foot to increase the membership in this organization by inviting members of other schools in the vicinity interested in the subjects to join.

I can best give you an idea of this organization by naming some of the topics discussed at its meetings: (*a*) History of Notation in Some of its Interesting Phases, (*b*) Short Cuts in Arithmetic, (*c*) Algebraic Fallacies, (*d*) Geometric Fallacies, (*e*) Famous Geometry Problems, (*f*) The Slide Rule, (*g*) Magic Squares, (*h*) Sector Compasses, (*i*) Sextant.

Other topics proposed are: (*k*) Interesting Facts from the Theory of Numbers, (*l*) Shadow Geometry, (*m*) Sun Dials.

Some of these topics occupy more than one meeting. Often

more than one pupil is scheduled to take part and all are urged and usually do take part in an active discussion of the topic in hand.

Our high school has an assembly two or three times a week. A committee of teachers and pupils arrange programs for these assemblies. It is the intention to have one of these assemblies each week conducted by a member or members of the school, teacher or pupils. Some such assemblies have been given in charge of a mathematics class or club.

Last year, for instance, the mathematics club gave an assembly on "Mathematics used in the Various Professions." This year, Columbus Day, the senior mathematics class gave an assembly in which they discussed the instruments used in navigation in the time of Columbus. These assemblies have proved interesting and instructive both to participants and auditors.

You know how dangerous it is to ask a person who has been abroad to tell you about the trip. I fear I have done much to show that it is just as dangerous to ask a person to tell something about his work. Hence, to avoid further trespassing on your patience I shall omit other details.

In brief, then, our high school course in mathematics with its fourfold aim of preparation for advanced work, training for accuracy, developing of scientific appreciation, and application to practical affairs, is in the hands of an organization department with an equipment which includes a library and incipient museum. Our course of study, two years' required and two years' elective work, is one in mathematics, not in algebra, geometry, etc., as distinct branches, thus keeping the broad point of view ever in the foreground. As devices to further our aims we require frequent summaries; we include field work with modern and ancient intruments; we lay stress on the real problems. We seek to stimulate the students' interests by encouraging a mathematics club and giving into the hands of pupils in the department some of our high school assemblies.

I am conscious of having included much that is commonplace. Perhaps to some of you, our system, as I have unfolded it, contains nothing new. I hope to many there will be at least one new idea to carry away.

Ethical Culture High School,
New York City.

THE NATIONAL GEOMETRY SYLLABUS COMMITTEE.

SUB-COMMITTEE ON LOGICAL CONSIDERATIONS.

BY EUGENE R. SMITH.

At the 1908 summer meeting of the National Educational Association the Mathematics Section authorized the appointment of a National Geometry Syllabus Committee, with Dr. H. E. Slaught as chairman. Lack of permanence of organization, and therefore of right to appropriate money, on the part of the section temporarily stopped the work, although Dr. Slaught continued to canvass the country for representative men to form the committee.

At the Baltimore meeting of the American Federation of Teachers of the Mathematical and the Natural Sciences, as the representative of the Middle States and Maryland Association, I presented the matter to the Federation, and urged the continuance of the work apparently dropped by the National Educational Association. The Federation at once adopted the scheme, accepted Dr. Slaught as chairman, and to avoid losing the work already done by him, asked him to nominate to the executive committee the teachers he thought should be on the committee. Soon afterwards the National Educational Association again took up the matter, and as a result, a joint committee of fifteen was appointed. This committee was then subdivided by the chairman into three sub-committees, of which I represent the one asked to report on the logical considerations of geometry. This committee is constituted as follows: David Eugene Smith, chairman; Eugene R. Smith, secretary; William Betz, of Rochester High School; C. L. Bouton, of Harvard, and William Fuller, of Mechanics Arts High School, Boston.

The work expected was outlined as follows: "The consideration of axioms, definitions (including new terms, symbols, distribution, etc.), assumptions, informal proofs; treatment of limits and incommensurables; time and place in the curriculum; purpose; historical notes; and other related topics.

The committee has considered all the points mentioned in the foregoing paragraph, and I will touch briefly on the details of the report that will be made to the general committee.

Axioms and Postulates.

The committee recommends the usual ones, with little modification. It does not approve of the attempt to reduce the assumptions of elementary geometry to a minimum nor does it believe that all the implicit assumptions referring to existence, betweenness, etc., are of advantage. A substitution axiom is recommended on account of its convenience in many proofs.

It may perhaps be of interest if the list of axioms and postulates is given entire.

Axioms.—(1) The equality axioms, including equals plus equals, equals minus equals, equals times equals, equals divided by equals, like powers and like roots of equals, things equal to the same thing are equal to each other. (2) The axioms relating to the use of unequals; including unequals plus and minus equals, times equals, and divided by equals, unequals plus unequals; unequals subtracted from equals; and an axiom showing that if a first quantity is greater than a second, a second than a third, etc., the first is greater than the last. (3) A substitution axiom. (4) The whole is greater than any of its parts, and is equal to the sum of all its parts.

Postulates.—(1) One straight line, and only one can be drawn through two given points. (2) A straight line may be produced to any required length. (3) A straight line is the shortest path between two points. (4) A circle may be described with any given point as a center and any given line segment as a radius. (5) A figure may be moved from one place to another, without altering its size or shape. (6) All straight angles are equal. (7) Through a given point one line and only one can be drawn parallel to a given line.

Terms.

The only new term directly recommended is "congruent," and this is now so well established as to be old rather than new. The terms "scholium" and "mixed line" are noted as obsolete and useless; the new terms "ray," "sect" and "transverse angles" are spoken of as not yet in common use, and teachers are left free to use them or not as they choose, with the idea that they will make their way if they are worthy of use. The two uses of the word "circle" are compared; the newer use of

"circle," meaning the line rather than the surface bounded, is recognized, but the committee takes the position that it is not as important that consistent use be made of either method of defining throughout geometry, as that the terms be so defined that the meaning is clear to the pupils.

Symbols.

Those in common use, including the ordinary algebraic signs, and the usual signs for the geometric figures are the only ones needed. New symbols should not be used unless they receive the sanction of the mathematical world. There is some question of a sign for "congruent," but the usage is so varied that the committee is not ready to recommend the adoption of any one. If I may express a personal opinion, I believe that the combination of the equal and the similar signs is the logical one for final adoption. The equal sign alone is too confusing on account of its various meanings, and the identity sign has a somewhat different significance, while, on the other hand, the combination of the equal and similar signs is used for nothing else, and expresses the double truth about congruence.

Definitions.

The simpler concepts are decided to gain nothing by the attempt to define them. Such concepts are point, line, surface, angle, space, etc.; this does not mean that such concepts should not be discussed as to their definite usage, but simply that, as there are no simpler terms by which to define them, pupils should not be asked to memorize formal definitions.

The easily defined terms that are used as a basis for propositions should be defined accurately, although the exact form of the definition is not material.

It is recommended that definitions that are not required until later parts of the geometry course be given when needed, although there is no objection to their being grouped in the text for ready reference.

Informal Proofs.

The committee recognizes the fact that many geometric truths are most easily taught without a formally worded proof. Many of the most evident facts follow almost directly from the defi-

nitions, and a word of explanation is all that is necessary. In such cases the understanding of the pupil is helped rather than hindered by the simplicity of the explanation, while the rigor of the development is not dangerously weakened. A few of these statements are:

If one straight line meets another the sum of the two adjacent angles is a straight angle, and conversely (and related propositions).

All straight angles are equal (if not used as a postulate).

Two straight lines can intersect in but one point.

If two angles are unequal, the greater angle has the less complement (and related propositions).

A straight line can cut a circle (circumference) in two points only.

Circles of equal radii are equal, and related statements.

A straight line can have but one point of bisection, and the related case for angles.

The bisectors of vertical angles lie in one straight line.

The bisectors of adjacent supplemental angles are perpendicular to each other.

All radii (and all diameters) of the same circle are equal.

A circle can have but one center.

Propositions relating to the conditions under which two circles (circumferences) intersect.

Polygons similar to the same polygon are similar to each other.

These propositions are, of course, only types of the ones to be proved informally; there are many others of a similar nature.

The committee believes that the experience of teachers both in this country, and even more in some of the European countries where this method is quite freely used, has shown that good results follow it when it is used in moderation. That it should be carried to an extreme would undoubtedly be dangerous and defeat its own purpose.

Limits and Incommensurables.

The committee takes the position that schools should not be required to teach this part of the subject, in other words, that it should not be considered a part of college preparation. On

the other hand, some explanation of the necessity for incommensurable case proofs is necessary and each teacher is left free to give to his pupils whatever they can assimilate, with the understanding that what is taught is for the purpose of giving a clearer grasp of the subject, and not as part of the required course. Many of the best schools have given up teaching this part of the subject some time ago, and it seems that this recommendation is only the formal statement of what is already becoming an accomplished condition.

Time and Place in the Curriculum.

For the ordinary secondary school, that is, high school or preparatory school, eight years of elementary school are presupposed. In the first year of the secondary school course algebra is usually taught, and the position of the committee is that geometry, except where a teacher believes in starting it with algebra and teaching them together, should follow in the second school year, and should have one and a half years for its completion. This means one and a half school years of five periods a week, the periods being the ordinary ones of approximately forty minutes. In this time it is desirable, if possible, to give the pupils some idea of the first part of solid geometry. This amount of time is perhaps more than is usually allowed, but it is none too much for a subject of the importance of geometry, and the adoption of this recommendation would do a great deal toward enabling teachers to get all the good possible out of the geometry course.

Purpose of Teaching Geometry.

There has recently been some attempt on the part of certain teachers to make the practical side of geometry dominate its teaching. This attempt is not a new thing, for similar movements have originated at various periods of its history, and with no important result. That genuine applications of any subject have a decided value goes without saying, but that the search for applications of, at least, doubtful character should carry teachers away from the real purpose of their teaching is a dangerous tendency. The committee welcomes anything that adds interest to the subject or broadens its outlook, but it feels

strongly that the chief value of the study of geometry arises from the fact that it is an exercise in logic, and logic of a kind concrete enough to be understood by the immature mind of a secondary school student. It therefore takes the position that the chief purpose of geometry is cultural, and deprecates any attempt to emasculate the subject in the search for so-called "utilitarian" applications.

The question as to whether a short course in practical geomtery would be wise in certain kinds of schools did not seem to come in the province of this committee. The committee would probably agree that for certain classes of pupils such a course would be wise. The position taken refers to the ordinary secondary school where the pupil presumably desires as broad a education as the time at his disposal allows.

Historical Notes.

The committee calls attention to the fact that historical notes about the various interesting features of the geometry add value and interest to the course. There are many sources from which such material can be obtained, and here again anything of a broadening character should be welcomed.

Solid Geometry.

The general principles already spoken of apply equally to solid geometry and little more need be said. The committee states the axioms and postulates that seem necessary, and suggests that latitude be allowed in the use of terms such as prismatic space and others in more or less use, while it does not specifically add any to the commonly used terms. The position taken in regard to limits and incommensurables holds also for solid geometry.

In regard to the purpose of the study of solid geometry, two are added to the one relating to the cultural side: to present a reasonable range of applications to mensuration; to cultivate the power of visualization, more especially of solid figures from two dimensional drawings.

As regards the first, it is of course evident that while plane geometry offers a comparatively small proportion of applicable propositions, solid geometry is used for many important meas-

urements, and so may fairly be said to have its applications to mensuration as an integral part of the purpose for which it is taught.

The cultivation of the power of visualization has probably not been given the recognition it deserves; and there can be no doubt that solid geometry possesses to a high degree the qualifications necessary to training this ability.

In conclusion I wish to say that this committee has tried to avoid any radical recommendations such as have not yet been approved by the mathematical world, while it has meant to be open minded toward improvements in geometry teaching. Whether it has succeeded in its aim is for teachers to judge from its report, but if the report is accepted as a sane and helpful one, then every teacher should work for its adoption as a standard, in spite of any personal preferences that may hinder his agreeing with all its details. It would be absolutely impossible for all teachers of geometry to agree as to all the details in its teaching; and it is fortunate that it is so, for otherwise there would be little progress, but it is possible for us to be ready to coöperate with each other in setting up a standard of uniformity. Even the members of this committee do not absolutely agree on all the details of its report, and this report is, as would of course be necessary, a consensus of opinion. If it shall appear to represent the consensus of opinion of all teachers of geometry, that for which the committee was appointed will have been accomplished. In making this plea for uniformity, I am thinking of the report of the entire committee as well as of the part that I represent, for perhaps the part in which uniformity will be of most value is the list of theorems being prepared by the second subcommittee.

POLYTECHNIC PREPARATORY SCHOOL,
BROOKLYN, N. Y.

ON THE CURRICULUM OF MATHEMATICS.*

By Isaac J. Schwatt.

It would appear that the processes of education have been carried on long enough to have established a fixed educational policy. The prevalent conflicting ideas and the present unrest in education indicate, however, that it is only in the beginning of its experimental stage and that educators are not, as yet, agreed on many of the most fundamental questions of their calling.

There are two markedly differing tendencies in educational methods today. One is to carry on instruction in such a manner as to make the knowledge gained appear practical; to supply the pupil with such information as, it is assumed, he can actually use in the ordinary pursuits of life. The other tendency is to emphasize the cultural value of knowledge with little regard to its utilitarian purpose.

In considering how much of the knowledge obtained at school a person actually uses in his daily work, we must not forget that the majority of men and women are artisans, mechanics, clerks, or followers of the many other occupations which are indispensable to the common good. I have failed to find, in any of the ordinary pursuits of life, the slightest trace of any direct application of the mathematics taught in the elementary schools, high schools, and colleges, beyond operations with simple fractions and comparatively small integers. Followers of the different occupations soon learn to perform these arithmetical operations with facility, and very often in a more effective manner than that which they have practiced at school. Even the engineer, greatly as it is to be deplored, is using more and more the ready-made data of the hand-books issued by manufacturing establishments. We can easily convince ourselves whether or not the mathematical disciplines find application in the daily pursuits of the average person. For such an in-

* Read at the annual meeting of the Association of Mathematical Teachers in New England, Boston, December 28, 1909.

quiry we do not need elaborately equipped laboratories, complicated instruments, or well-stocked libraries. The members of our own family, our neighbors, any person we meet on the street and who may be willing to give us the information we ask, contribute to prove that mathematics has little application in the life of the average person. It is true that the principles of geometry enable us to measure the height of a tree if the sun shines brightly and the tree is not obstructed by surrounding trees, and if we have a long enough pole at our disposal; that the game of base ball, of pool and of many others are based on geometrical and mechanical principles. But no one would suggest that geometry shall be studied for such and similar purposes in life.

The purpose of the study of any mathematical discipline is the development of the mind of the learner. The principles of mathematics do not contain even the interesting information of many of the other subjects of the curriculum. It is immaterial to the average person whether the sum of the angles of a triangle is two right angles, or more or less. Unless a person is benefited by the mental training which the study of such theorems is capable of giving, the knowledge and information gained must be considered a waste of time and energy to him. If mathematics cannot help the learner to gain power of mind, I for one am heartily in accord with those educators who suggest that the study of mathematics be eliminated from the secondary schools.

It is extremely unfortunate that there is a tendency to force into some of the subjects of the curriculum a quality that they do not naturally possess. This is bound to influence in a pernicious manner the teaching of these subjects. It is held that all studies have the quality of developing power of mind; but, if a subject be taught with reference to its practical application, it may happen that neither the one nor the other purpose of education will be accomplished. Any work that the pupil does in a subject because it may find application in life, tends only to develop and strengthen utilitarian and mercenary qualities in him, and the desire of immediate reward and gain.

We are prone to judge of circumstances and conditions from those immediately surrounding us. This is the reason why espe-

cially in more intelligent circles, there is not attached to education as a means for developing mental faculties and for increasing common sense, as much importance as it deserves. Yet if the normal equilibrium of society is disturbed we soon find that most people are not always able to exercise the best judgment in governing their affairs. Education is in fact the only hope for bettering conditions.

Considering the manifold and varied occupations and professions necessary to meet all the needs of modern society, it is manifestly impossible for the school to teach, even superficially, all the subjects relating to these needs, and at the same time develop in the student the moral and intellectual powers that are indispensable to true success in life. Whatever the occupation of a person, he is in need of character and the powers of mind that are awakened, developed, and strengthened by a proper education.

There are many things which a person must know to take his position in the world. There is much knowledge the acquisition of which adds greatly to the enjoyment of life. We should know the reasons for the phenomena occurring about us. One who does not understand the workings of nature cannot participate in the greatest enjoyment bestowed upon mankind. There are also questions pertaining to economic conditions which every citizen should thoroughly understand.

The school should develop and cultivate in the young a taste for good music. While listening to it, the noblest instincts are aroused, and the most tender emotions are awakened in those who have acquired an understanding of it. It is desirable that the school should train the eye and the ear, and develop the various latent faculties in the young. The school should train the pupil to be handy with tools, a desirable accomplishment in every household. But mental development should not be sacrificed to the acquirement of manual skill. Since the young especially find mental effort the hardest task, they readily take to manual work which calls for little mental application rather than to such studies as require concentrated thought.

It is our belief that good health, good character, and power

of mind are the best assets with which to start life. With these qualifications the youth will be better equipped for any work and be more successful in the true sense of the word than if he has learned how to make a candlestick when he is to become a banking clerk. Mechanics and artisans who possess great skill are greatly handicapped in their work if they do not have the power of mind to apply their skill in an advantageous manner.

Since the school cannot teach all the subjects which may be useful or interesting in life, it must select those which will serve best the true purposes of education, which will awaken, develop and strengthen those qualities which we all must possess, whatever our occupation. These, as stated before, are character and power of mind. Just as it requires different ingredients to make up the diet necessary to keep all the bodily organs healthy, strong, and in a harmonious state of development, so the mind needs variety of training for its proper expansion. For the curriculum such subjects must be selected as, blended together, will afford the best means for the development of character and mind. In making such a selection, special attention should be paid to the student's age and to the state of his mental progress. There is a tendency to extend the elective system, now in vogue in some of the colleges, to the secondary schools, that is, to select for the individual student such subjects as seem best fitted to his state and to his personal preference. Such a course ought to be pursued when the student is choosing his life work and not while he is getting a general education. If a selection of subjects is advisable, the courses which in the secondary school or in the college the student ought to choose, must not be those for which he is best fitted, or which he can easily learn, or for which he has a special taste and inclination, but they must be such as will develop in him those desirable faculties which he lacks, or those qualities of mind which in him are less strong than others. In this way the student will approximate to a perfect man or woman. It ought to be the duty of the teacher to study the faculties, abilities and capabilities of the student in order that he may be able to advise him as to what occupation or profession he is best fitted for. The importance of the teacher's services in this respect cannot be overestimated.

Parents are often inclined to leave the choice of a profession or occupation with their children. Now the young are, as a rule, attracted to an occupation more by exterior appearances or by the promise of lucrativeness than by their fitness for it. It is preposterous to allow a young man of sixteen or eighteen with no knowledge of the world and the things in it, to choose for himself his life work. There are men and women, who, although they perform the duties of their present occupation satisfactorily, might be better fitted for some other occupation and in which they could do more good to themselves and to their fellowmen.

In a properly arranged scheme of studies an undue amount of time and energy should not be devoted to any one subject at the expense of another. A person trained, for instance, in mathematics without having enjoyed the benefit to the mind resulting from other studies, would be in a similar position to the newly arrived Finnish girl of whom the following is related: A lady who wished to engage her services tried, through an interpreter, to find out what her accomplishments were. Having been brought up in a rather crude civilization, the girl was naturally unfamiliar with the requirements of a well-to-do American household. The lady, having formerly had favorable experiences with foreign help, pressed the interpreter to discover some accomplishment which the girl might possess. The interpreter finally joyfully reported that the girl could milk a reindeer.

Power of mind can be acquired from clear ideas only. Character can be developed by exercising and practicing the virtues which constitute it. The purpose of education is to accomplish both. I was greatly interested in the preliminary report issued a few weeks ago by the United States Immigration Commission. This report which was prepared with the assistance of expert anthropologists, shows conclusively that the physical form of the descendents of some of the European immigrants is different from that of their ancestors. Children born a few years after the arrival of their parents in this country, develop a different form of head from that of their parents, notwithstanding that the head has always been considered one of the most persistent hereditary features of a race. This goes to

show that the individual is influenced by environment to a much greater extent than has been supposed. Just as a change in economical, social and climatic conditions produces a change in the physical form of the individual, so education should produce a change in the character and mind of the pupil. Education will accomplish this, if its work be as thorough, as steady, and effective, as the ever present influence of environment.

Education depends on a variable, the requirements of life. It depends not only on the magnitude but also on the intensity of this variable. The purpose of education now is the same as it was centuries ago. But as the demands of life are so much greater today than in former years when temptations were few and competition was mild, the intensity of education must be correspondingly increased.

I feel that the curriculum of our high schools is, on the whole, modeled after the course of study of some of the foreign gymnasiums, more especially after the German. But the conditions in this country differ widely from those in foreign nations. Even today in some of these, very few enter these schools, except those intending to take up a profession. I have examined the programs (for the year 1908) of some widely scattered gymnasiums and I have found that nearly all the graduates have indicated their intention to take up a profession. In this country, however, a very small percentage of high school pupils continue their studies in a university and a large percentage of these take liberal college courses only.

Since the opportunity of attending high school is enjoyed by comparatively few, and still fewer have the privilege of attending college, it is the duty of the higher schools to see that all their students shall gain such qualifications as will fit them for true leadership among their fellowmen. They shall be missionaries for all that is right and just, and good and noble.

In this connection there is a question which I have not been able to answer. Since the high schools and colleges educate only a very small percentage of the youths of the country, shall the college attract only the ablest and the most promising youths and prepare them for true leadership in the sciences, in the arts, and in all activities of life; or shall the college admit all those who are sufficiently prepared to pass the en-

trance examinations? Under the present system, however, it is the duty of the teacher to pay especial attention to those students who are least gifted and who need our help most.

There is no doubt that, on the whole, our teaching has not been effective in imparting to our students perfectly clear ideas of the mathematical subjects taught. In addition to those causes for the failure of our efforts which I have discussed before the Association of Teachers of Mathematics of the Middle States and Maryland, I believe that some of the subject matter in mathematics ordinarily taken up in elementary and secondary schools is beyond the age and capability of the average pupil. He may acquire some proficiency in performing operations but, as a rule, is not capable of attaining a perfectly clear understanding of the principles and ideas underlying them.

In education quality is the only decisive factor. Unless the pupil acquires perfectly clear ideas of every point of the ground covered, the instruction must be considered a failure and a waste of time and energy. One clear idea which is entirely the pupil's own mental possession is of infinitely greater benefit to his development, than if he were to go over a great amount of ground without gaining a thorough understanding of it. Some of us may recall that we were not always able to gain a perfectly clear conception of the principles and ideas we studied at school. Sometimes it was because we were not allowed sufficient time for the ideas to grow on us, so to speak. More often, however, it was the state of our mental development, even though it was normal for our age. It is not possible for the average high school pupil to gain perfectly clear ideas of each subject of secondary mathematics and to derive the mental benefit which their study is capable of yielding. It takes greater maturity than is possessed by the average high school pupil to thoroughly understand some of the principles of demonstrative geometry, for instance, and to acquire facility in solving constructions. On that account, if in the high school with a four years' course mathematics is taught only during a part of the time, it ought to be taken up during the last years, when the student is more mature and is better prepared to follow the instruction.

(*To be continued.*)

NEW BOOKS.

The Ethics of Jesus. By HENRY CHURCHILL KING. New York: The Macmillan Co. Pp. 293. $1.50 net.

The object of this book is to so unfold the teachings of Jesus and allow them to speak for themselves that the reader may get a better and clearer conception of him as a teacher of morals and as a guide in conduct. Knowing the teachings we are to become doers and not hearers only. It is interesting reading and an invaluable book for teachers.

Attention and Interest. By FELIX ARNOLD. New York: The Macmillan Co. Pp. 269. $1.10 net.

This study in psychology and education is an attempt to clarify and arrange the many facts brought out by recent experiments. The reader is left free to draw his own conclusions and theories from the data given.

Manual of Gardening. By L. H. BAILEY. New York: The Macmillan Co. Pp. 539. $2.00 net.

This volume is a combination and revision of the main parts of "Garden-Making" and "Practical Garden-Book" by the same author, together with much new material. It is written for the home-maker and teachers will find it not only interesting but valuable as a guide in beautifying home and school grounds. It also treats of fruit and vegetable gardening and will be found helpful to many in these directions.

How to Keep Hens for Profit. By C. S. VALENTINE. New York: The Macmillan Co. Pp. 289. $1.50 net.

A considerable portion of the material in this book was first published in the *New York Tribune Farmer*. Much new material has been added and the whole unified and brought down to date. It is coming to be more and more realized that the hen, with proper care, is a very profitable animal and such works as this will do much to acquaint those who keep hens with the meaning of proper care and attention.

The Principles of Education. By W. C. RUDIGER. Boston: Houghton, Mifflin and Company. Pp. 305.

The object of this book as stated by the author is "to bring together and organize the leading tendencies in modern educational thought pertaining to the bases, aims, values, and essential content of education." The biological theory is accepted as the general guide and "adjustment" as the aim of education. The book is well written and will be helpful in showing the rapid transition that is being made in educational theory.

Rara Arithmetica. By DAVID EUGENE SMITH. Boston: Ginn and Company. Pp. 507. $4.50.

This volume is a catalogue of the arithmetics written before 1601 with a description of those in the library of Mr. George A. Plimpton. It is

very profusely illustrated with facsimiles of pages, many of which are of title pages of rare first editions. It is a work of large magnitude and importance and has been done in a manner to reflect credit on the author.

NOTES AND NEWS.

The fourteenth meeting of the association was held at George Washington University on March 26. The program opened with an address of welcome by Dr. Charles W. Needham, president of George Washington University. After the routine business had been gone through with, the secretary read a motion that was recommended to the association by the council. It was necessitated by the requirements of the post office authorities in regard to second-class mail matter. The motion was as follows:

In consideration of the requirements of the post office authorities as to the entering of THE MATHEMATICS TEACHER as second-class mail matter, and since the sections of this association have passed motions approved the following procedure, it is hereby moved that the constitution of this association shall be amended so that the annual dues shall be fifty cents to those members who subscribed for THE MATHEMATICS TEACHER, and sixty cents to those members who do not so subscribe, the subscription to THE MATHEMATICS TEACHER being an entirely separate matter. It is further moved that this shall serve as notice of intention to so amend the constitution.

The motion was unanimously passed. The price of THE MATHEMATICS TEACHER has been fixed, for the present, at fifty cents a year to members, and one dollar a year to non-members.

The following papers were read and discussed:

"Teachers' Salaries and the Cost of Living," L. D. Arnett, Library Division, Bureau of Education.

"Some Remarks on Approximate Computation," M. J. Babb, University of Pennsylvania.

"Special Devices Used in the Teaching of Mathematics" (ten-minute papers). (*a*) Algebra, Eugene Randolph Smith, Polytechnic Preparatory School, Brooklyn, N. Y.; (*b*) Plane Geometry, Paul N. Peck, George Washington University, Washington, D. C.; (*c*) Solid Geometry, Howard F. Hart, High

School, Montclair, N. J.; (*d*) Trigonometry, William H. Jackson, Haverford College, Haverford, Pa.

As the chairman of the Algebra Syllabus Committee was unable to be present, the secretary gave a report of progress for the committee, and requested a vote on the inclusion of certain topics. The meeting voted not to include in advanced algebra the following topics:

1. General proof of the Binomial Theorem.
2. Convergency of Series.
3. Exponential Theorem and Logarithmic Series.
4. Variation.

New Members.

N. Y. Candle, Catherine Patricia, A.B.; Normal College, New York. 208 East 79th St.

N. Y. Capron, Florence Elizabeth, A.B.; High School, Jersey City, N. J. 502 W. 149th St., New York City.

N. Y. Collins, Hamlet Paul, B.Sc.; High School, Jersey City, N. J. 47 Bryant Terrace.

Deal, Alice, A.B.; McKinley High School, Washington, D. C. The Victoria.

Farr, Sarah M.; Central High School, Washington, D. C. 1347 Newton St.

S. Gantt, Lillian, A.B.; High School, Amsterdam N. Y. 5 Kimball St.

S. Lewis, Mabel E., A.B.; High School, Greenport, Long Island, N. Y.

Little, William Darlington, A.M.; High School, Jersey City, N. J. 550 Summit Ave.

Warner, Frank Bradford, A.B.; The Hoover School. Patterson, N. J.

S. Ahern, Katherine C., A.B.; High School, Amsterdam, N. Y. 46 Church St.

S. Atwell, Robert King, A.B.; Syracuse University, Syracuse, N. Y.

S. Avery, Royal A., Ph.B.; North High School, Syracuse, N. Y. 1317 First North St.

P. Hanford, Bessie Eugenia, A.B.; Wilson College, Chambersburg, Pa.

Eugene R. Smith,
Secretary.

Fifty members of the Philadelphia Section attended the annual dinner held at the Bartram on April 9. Chairman George Alvin Snook presided, introducing the after-dinner speakers with happy comment.

President W. H. Metzler read a thoughtful convincing paper on "Formal Discipline."

Dr. Lucy Langdon Williams Wilson, of the Philadelphia Normal School, made clear the benefits which would result from a simple course in arithmetic in the elementary schools. First, the student could acquire greater accuracy in handling numbers; second, he could devote more time to the study of the world in which he lives; third, he would bring to his high school work a freshness and interest now lost in the mechanical grind of the early years.

Dr. Cheesman A. Herrick, president of Girard College, and Dr. Isaac T. Schwatt, of the University of Pennsylvania, each ably and earnestly contended for the faith as it had been delivered to him, Dr. Herrick emphasizing the vocational and Dr. Schwatt the cultural value of mathematics.

The committee in charge of the dinner were Mr. Walter Roberts, Miss Carolyn Wood Stretch and Mr. T. Eugene Walker.

THE special topic of the spring meeting of the Philadelphia Section was "The Locus Problem." Dr. Fletcher Durell, of the Lawrenceville School, introduced the topic by speaking particularly of the scope of the locus. He was followed by Dr. H. C. Whitaker, of the Southern High School, who presented in addition to the more common applications, some interesting practical illustrations that were new to many present. A general discussion followed.

The following officers were elected for the ensuing year: *Chairman,* Dr. George H. Hallett, University of Pennsylvania; *Vice-chairman,* Edward D. Fitch, Delancey School; *Secretary,* Elizabeth B. Albrecht, Philadelphia High School for Girls; *Members of the Executive Committee,* Agnes H. Long, Wm. Penn High School for Girls; Harry P. Rothermel, Southern High School.

MISS MURIEL SMITH, a graduate student of Rochester University, was elected to the mathematical staff of the William Penn High School as a result of a competitive examination. In a similar manner, Miss Florence Rothermel, formerly of the Camden and the Pottstown High Schools, was awarded a posi-

tion in the Philadelphia High School for Girls. Both teachers are members of our association.

MR. WILLIAM BARRETT, formerly head of the mathematical department of the Friends' Central School, has recently been promoted to the principalship of the school. Mr. Barrett was formerly the secretary of the Philadelphia Section, and has always taken an active part in the work of the section.

THE eighth regular meeting of the Rochester Section was held at the University of Rochester on March 5, with an attendance of twenty-four. The following papers were presented:

"Suggestions on the Teaching of Elementary Algebra," Mr. F. L. Lamson, The University of Rochester.

"Mathematics in French Schools," Professor W. E. Etzel, St. Bernard's Seminary, Rochester.

"The Place of Industrial Training in the American School," Mr. A. P. Fletcher, Supervisor of Manual Training, Rochester Public Schools.

"Examples of Common Loci," Mr. C. W. Watkeys, The University of Rochester.

Professor Etzel's remarks were especially interesting from the fact that he taught mathematics in France for many years.

It was voted that the sentiment of the Rochester Section is that the association should adopt the proposed separation of the annual dues and the subscription to the THE MATHEMATICS TEACHER.

Five applications for membership were received.

THE New York Section held its second regular meeting for the year at the High School of Commerce, Friday, March 11, 1910, at 8 p. m. The following program proved of interest: Report of Secretary-Treasurer. Announcement of the Committee for the further study of the problem of Industrial Mathematics, Mr. W. E. Breckenridge, chairman. General Topic for the Evening: "The National Geometry Syllabus."

As the reader is doubtless aware, a committee known as the National Geometry Syllabus Committee was appointed about a year ago as a joint committee of the National Educational Association and the Federation of Teachers of the Mathematical and the Natural Sciences, to prepare a "National

Syllabus of Geometry." The committee was divided into three sub-committees, each of which will be represented on our program as follows:

(*a*) Sub-committee on Logical Considerations, Mr. Eugene R. Smith, head of department of mathematics, Polytechnic Preparatory School, Brooklyn.

(*b*) Sub-committee on Lists of Basal Theorems, Dr. Herbert E. Hawkes, professor of mathematics, Yale University, New Haven, Conn.

(*c*) Sub-committee on Exercises and Applications, Mr. Clifford B. Upton, department of mathematics, Teachers College, Columbia University.

General discussion.

TEACHERS wishing to secure a copy of the recent monograph on "The Historical Development of the Decimal Fraction," by Professor David Eugene Smith, may do so by addressing the secretary of Teachers College, Columbia University.